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From dispersionless to soliton systems via Weyl–Moyal-like deformations

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Abstract

The formalism of quantization deformation is reviewed and the Weyl–Moyal-like deformation is applied to systematic construction of the field and lattice integrable soliton systems from Poisson algebras of dispersionless systems.

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1. Introduction

Recently, various aspects of the Moyal deformation theory and its application to the integrable field systems, which lead to the so-called Moyal-type Lax dynamics, have become of increasing interest [1–5]. The aim of this paper is to present a complete scenario of construction of the field and lattice soliton systems by Weyl–Moyal-like deformations from Poisson algebras of underlying dispersionless systems. The Weyl–Moyal-like deformation is a special case of the deformation quantization.

In the theory of evolutionary systems (dynamical systems) one of the most important issues is a systematic method for construction of integrable systems. As integrable systems we understand those which have an infinite hierarchy of symmetries and conservation laws. It is well known that a very powerful tool, called the classical R-matrix formalism, proved to be very fruitful in systematic construction of the field and lattice soliton systems as well as dispersionless systems (see [6–18] and the references there).

The crucial point of the formalism is the observation that integrable dynamical systems can be obtained from the Lax equations

$$L_t = \mathrm{ad}^*_A L = [A, L] \tag{1.1}$$

i.e. a coadjoint action of some Lie algebra $\mathfrak{g}$ on its dual $\mathfrak{g}^*$, with the Lax operators taking values from this Lie algebra $\mathfrak{g}^* \cong \mathfrak{g}$, equipped with the Lie bracket $[\cdot\,,\cdot]$. From (1.1) it is clear that we confine to such algebras $\mathfrak{g}$ for which the dual $\mathfrak{g}^*$, related to $\mathfrak{g}$ by the duality map $\langle\cdot\,,\cdot\rangle \to \mathbb{R}$, can be identified with $\mathfrak{g}$. So, we assume the existence of a scalar product $(\cdot\,,\cdot)_{\mathfrak{g}}$ on $\mathfrak{g}$ which is symmetric, non-degenerate and ad-invariant

$$(\mathrm{ad}_a b, c)_{\mathfrak{g}} + (b, \mathrm{ad}_a c)_{\mathfrak{g}} = 0. \tag{1.2}$$

This abstract representation (1.1) of integrable systems is referred to as the Lax dynamics.

On the space of smooth functions on the dual algebra $\mathfrak{g}^*$ there exists a natural Lie–Poisson bracket

$$\{H, F\}(L) := \langle L, [\mathrm{d}F, \mathrm{d}H]\rangle \qquad L \in \mathfrak{g}^* \qquad H, F \in \mathcal{C}^\infty(\mathfrak{g}^*) \tag{1.3}$$

where $\mathrm{d}F,\ \mathrm{d}H$ are differentials belonging to $\mathfrak{g}$. A linear map $R : \mathfrak{g} \to \mathfrak{g}$, such that the bracket

$$[a, b]_R := [Ra, b] + [a, Rb] \tag{1.4}$$

is a second Lie product on $\mathfrak{g}$ is called the classical R-matrix. A sufficient condition for R to be an R-matrix is

$$[Ra, Rb] - R[a, b]_R + \alpha[a, b] = 0 \qquad a, b \in \mathfrak{g} \tag{1.5}$$

where α is some real number, called the Yang–Baxter equation YB(α).

Then, bracket (1.4) is related to another Lie–Poisson bracket and the appropriate Poisson tensor is as follows:

$$\{H, F\}_1(L) := \langle L, [\mathrm{d}F,\ \mathrm{d}H]_R\rangle =: \langle \mathrm{d}F, \theta_1(L)\,\mathrm{d}H\rangle. \tag{1.6}$$

The Casimir functions C of the natural Lie–Poisson bracket (1.3), i.e.

$$\{C, F\} = 0 \qquad \forall F \in \mathcal{C}^\infty(\mathfrak{g}^*) \tag{1.7}$$

are in involution with respect to the Lie–Poisson bracket (1.6). Hence, the vector fields generated by such Casimir functions

$$L_{t_n} = \theta_1(L)\,\mathrm{d}C_n = [R\,(\mathrm{d}C_n)\,, L] \tag{1.8}$$

commute mutually as the map $\theta \circ \mathrm{d}$ is a Lie algebra homomorphism. Moreover (1.8) are Hamiltonian equations. The hierarchy of evolution equations (1.8) is the Lax hierarchy with common infinite set of symmetries and conserved quantities. In this sense (1.8) represents a hierarchy of integrable evolution equations.

It is known that the systems (1.8) are tri-Hamiltonian with respect to three Poisson brackets called the linear, quadratic and cubic, reflecting the dependence on the L. The linear tensor $\theta_1(L)$ takes the form [7]

$$\theta_1(L)\,\mathrm{d}H = -\mathrm{ad}_L R(\mathrm{d}H) - R^*\mathrm{ad}_L\,\mathrm{d}H \tag{1.9}$$

where R^* is the adjoint of R, i.e. $(Ra, b)_{\mathfrak{g}} = (a, R^*b)_{\mathfrak{g}}$. The quadratic case is more complex. A tensor $\theta_2(L)$ [12]

$$\theta_2(L)\,\mathrm{d}H = A_1(L\,\mathrm{d}H)L - LA_2(\mathrm{d}H\,L) + S(\mathrm{d}H\,L)L - LS^*(L\,\mathrm{d}H) \tag{1.10}$$

defines a Poisson tensor if the linear maps $A_{1,2} : \mathfrak{g} \longrightarrow \mathfrak{g}$ are skew-symmetric solutions of the YB(α) (1.5), where $\alpha \neq 0$, and the linear map $S : \mathfrak{g} \longrightarrow \mathfrak{g}$ with adjoint S^* satisfies

$$\begin{aligned} &S\,([A_2a, b] + [a, A_2b]) = [Sa, Sb] \\ &S^*\,([A_1a, b] + [a, A_1b]) = [S^*a, S^*b]. \end{aligned} \tag{1.11}$$

In the special case when $\frac{1}{2}(R - R^*)$ satisfies the YB(α), for the same α as R, under the substitution $A_1 = A_2 = R - R^*$, $S = S^* = R + R^*$, the quadratic Poisson operator (1.10) reduces to [7]

$$\theta_2(L)\,\mathrm{d}H = -\mathrm{ad}_L R\,\mathrm{ad}_L^+\,\mathrm{d}H - LR^*\mathrm{ad}_L\,\mathrm{d}H - R^*\,(\mathrm{ad}_L\,\mathrm{d}H)\,L \tag{1.12}$$

where $\mathrm{ad}^+_L A = LA + AL$, and conditions (1.11) are equivalent to $\mathrm{YB}(\alpha)$ for R. Another special case is when the maps $A_{1,2}$ and S originate from decomposition of the R-matrix (1.17)

$$R = A_1 + S = A_2 + S^* \tag{1.13}$$

where $A_{1,2}$ are skew-symmetric. Then the sufficient condition for the Poisson property of θ_2 is [14]

$$[A_{1,2}a, A_{1,2}b] + [a, b] = A_{1,2}([A_{1,2}a, b] + [a, A_{1,2}b]). \tag{1.14}$$

Finally, the cubic tensor θ_3 takes the form

$$\theta_3(L)\,\mathrm{d}H = -\mathrm{ad}_L R(L\,\mathrm{d}HL) - LR^*(\mathrm{ad}_L\,\mathrm{d}H)L. \tag{1.15}$$

For constructing the simplest R-structure let us assume that the Lie algebra $\mathfrak{g}$ can be split into a direct sum of Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$, i.e.

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \qquad [\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm. \tag{1.16}$$

Denoting the projections onto these subalgebras by $P_\pm$, we define the R-matrix as

$$R = \tfrac{1}{2}(P_+ - P_-). \tag{1.17}$$

It is easy to verify that (1.17) solves $\mathrm{YB}\left(\frac{1}{4}\right)$.

Following the above scheme, we are able to construct in a systematic way integrable Hamiltonian systems, with infinite hierarchy of involutive constants of motion and infinite hierarchy of related commuting symmetries, once we fix a Lie algebra. For example, the Lie algebra of pseudo-differential operators with the commutator leads to the construction of soliton systems [6–11]. The Lie algebra of shift operators leads to lattice field systems [12–14]. On the other hand, the Poisson algebras (which are Lie algebras with associative, commutative multiplication) of formal Laurent series lead to the construction of dispersionless systems [15–18].

As well known, a quasi-classical limit of field and lattice soliton systems gives related integrable dispersionless systems. We would like to invert this procedure and construct field and lattice soliton systems from some classes of integrable dispersionless systems through a Weyl–Moyal-like deformation quantization procedure. Actually, we will do it on the level of their Lax representations.

The idea behind the deformation quantization theory [19–22] is that a classical system can be obtained from a quantum system by the quasi-classical limit $\hbar \to 0$, where $\hbar$ is the Planck constant divided by 2π. Therefore, the quantization of classical systems should be done by appropriate deformations depending on a formal parameter $\hbar$. The classical fields (observables) belong to the associative commutative algebra of smooth functions, with standard multiplication, equipped with the Poisson bracket $\{\cdot, \cdot\}$. The idea of deformation quantization relies on the deformation of the usual multiplication to the new associative but non-commutative product called $\star$-product. It depends on the formal parameter $\hbar$, with the assumption that the $\star$-product in the limit $\hbar \to 0$ reduces to the standard multiplication and also that the Lie bracket $\{f, g\}_\star := \frac{1}{\hbar}(f \star g - g \star f)$, where f, g are smooth functions, reduces to the Poisson bracket. As well known, an arbitrary Poisson tensor, corresponding to the Poisson bracket, can be written by the wedge product of appropriate commuting vector fields. Then, the $\star$-product can be easily constructed by the so-called Weyl–Moyal-like deformation. The details will be given in the next section.

This paper is organized as follows. In section 1 we briefly present a number of basic facts and definitions concerning the classical R-matrix formalism. In section 2 we review the deformation quantization theory and present the Weyl–Moyal-like deformations. The Poisson algebras of formal Laurent series are introduced in section 3 and then, in section 4, the Weyl–Moyal-like deformation procedure is applied to them. In section 5, we apply the formalism of R-matrix to the quantized Poisson algebras and illustrate the results with particular examples. Finally, in section 6 are given some conclusions.

2. Star products and deformation quantizable Poisson brackets

Let $\mathcal{A} = \mathcal{C}^{\infty}(M)$ be the space of all smooth ($\mathbb{R}$ or $\mathbb{C}$ valued) functions on $2n$-dimensional smooth manifold M. Let $\{\cdot,\cdot\}_{PB}$ be the classical Poisson bracket, which is bilinear, skew-symmetric and satisfies the Jacobi identity. Obviously, $\mathcal{A}$ is a commutative, associative algebra over $\mathbb{R}$ or $\mathbb{C}$ with the standard multiplication.

Let $\star$ be the deformed associative non-commutative multiplication on $\mathcal{A}$ given by the following formula:

$$f \star g = \sum_{k\geqslant 0} \hbar^k B_k(f,g) \qquad f,g \in \mathcal{A} \tag{2.1}$$

where $\hbar$ is the formal parameter and $B_k : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ are bidifferential (bilinear) operators. We also define deformed bracket as a commutator

$$\{f,g\}_\star := \frac{1}{\hbar}(f\star g - g\star f). \tag{2.2}$$

Definition 2.1. *An associative deformed multiplication $\star$, given by the formula (2.1), is a formal quantization of the algebra $\mathcal{A}$ and is called the $\star$-product if*

(a) $\lim_{\hbar\to 0} f\star g = fg$
(b) $c\star f = f\star c = cf \qquad c\in\mathbb{R},\mathbb{C}$
(c) $\lim_{\hbar\to 0}\{f,g\}_\star = \{f,g\}_{PB}$.

Lemma 2.2. *The bracket (2.2) defined by the $\star$-product is bilinear, skew-symmetric and satisfies the Jacobi identity. So, it is a well-defined Lie bracket.*

The proof is obvious as the Jacobi identity is a consequence of an associativity of multiplication $\star$. Hence, bracket (2.2) is called the deformation quantization of the underlying classical Poisson bracket $\{\cdot\,,\cdot\}_{PB}$.

As follows from definition (2.1):

$$B_0(f,g) = fg \tag{2.3}$$

and

$$B_1(f,g) - B_1(g,f) = \{f,g\}_{PB}\,. \tag{2.4}$$

The associativity of the $\star$-product implies that the bilinear maps B_k satisfy the equations

$$\sum_{s=0}^{k}[B_s(B_{k-s}(f,g),h) - B_s(f,B_{s-k}(g,h))] = 0 \qquad k\geqslant 1. \tag{2.5}$$

Hence, B_1 satisfies the equation

$$B_1(f,g)h - fB_1(g,h) + B_1(fg,h) - B_1(f,gh) = 0. \tag{2.6}$$

Let $D:\mathcal{A}\longrightarrow\mathcal{A}$ be a linear automorphism parametrized by $\hbar$, such that

$$Df = \sum_{k\geqslant 0}\hbar^k D_k f \qquad D_0 = 1 \tag{2.7}$$

where D_k are differential operators. Such an automorphism produces a new $\star'$ in $\mathcal{A}$ in the following way:

$$f \star' g := D(D^{-1} f \star D^{-1} g). \tag{2.8}$$

The associativity of the new $\star'$ follows from the associativity of the old $\star$-product, as

$$\begin{aligned} f \star' (g \star' h) &= f \star' D(D^{-1} g \star D^{-1} h) = D(D^{-1} f \star (D^{-1} g \star D^{-1} h)) \\ &= D((D^{-1} f \star D^{-1} g) \star D^{-1} h) = D(D^{-1} f \star D^{-1} g) \star' h = (f \star' g) \star' h. \end{aligned} \tag{2.9}$$

By transformation (2.7) one finds the following expression:

$$B_1'(f, g) = B_1(f, g) - f D_1 g - D_1 g \cdot f + D_1(fg). \tag{2.10}$$

Then

$$B_1'(f, g) - B_1'(g, f) = B_1(f, g) - B_1(g, f) = \{f, g\}_{PB} \tag{2.11}$$

and

$$\lim_{\hbar \to 0} \{f, g\}_{\star'} = \lim_{\hbar \to 0} \{f, g\}_{\star} = \{f, g\}_{PB} . \tag{2.12}$$

Hence, the new $\star'$ is the second well-defined $\star$-product on $\mathcal{A}$.

Definition 2.3. *Two $\star$-products: $\star$ and $\star'$ are called gauge equivalent or simply equivalent if there exists a linear automorphism $D : \mathcal{A} \longrightarrow \mathcal{A}$ (2.7) such that (2.8).*

Let us now consider a Weyl–Moyal-like deformation. It is well known that an arbitrary classical Poisson bracket can be presented in the following form:

$$\begin{aligned} \{f, g\}_{PB} &= f \left(\sum_{i=1}^{n} Y_i \wedge X_i \right) g = f \left(\sum_{i=1}^{n} (Y_i \otimes X_i - X_i \otimes Y_i) \right) g \\ &= \sum_{i=1}^{n} [Y_i(f) X_i(g) - X_i(f) Y_i(g)] \end{aligned} \tag{2.13}$$

where $X_i, Y_i, i = 1, \ldots, n$ are pair-wise commuting vector fields on $2n$-dimensional smooth manifold M and $f, g \in \mathcal{A} = \mathcal{C}^\infty(M)$. The Jacobi identity for (2.13) follows from the commutativity of vector fields X_i, Y_i. From relation (2.4) there are two natural deformations of the classical bracket (2.13) induced by

$$B_1 = \frac{1}{2} \sum_{i=1}^{n} Y_i \wedge X_i \tag{2.14}$$

and by

$$B_1' = \sum_{i=1}^{n} Y_i \otimes X_i \tag{2.15}$$

respectively. In what follows, we will use the Einstein summation convention in the case of repeating indices i, j at the vectors X, Y and a standard convention (with the summation symbols) otherwise. The first case (2.14) leads to the Weyl–Moyal-like deformed multiplication

$$f \star g = f \exp\left[\frac{\hbar}{2} Y_i \wedge X_i \right] g. \tag{2.16}$$

If the classical Poisson bracket (2.13) is a canonical one, i.e. $Y_i = \partial_{p_i}, X_i = \partial_{x_i} (\partial_x = \partial/\partial x, \partial_p = \partial/\partial p)$, then product (2.16) is the Groenewold product [23] and the deformed bracket (2.2) is the well-known Moyal bracket [24]. Expanding (2.16) one finds

$$\begin{aligned} f \star g &= \sum_{s=0}^{\infty} \frac{\hbar^s}{2^s s!} f \prod_{k=1}^{k=s} Y_{i_k} \wedge X_{i_k} g \\ &= \sum_{s=0}^{\infty} \frac{\hbar^s}{2^s s!} \sum_{m=0}^{s} (-1)^m \binom{s}{m} \left(Y_{i_1} \ldots Y_{i_{s-m}} X_{j_1} \ldots X_{j_m} f\right) \left(Y_{j_1} \ldots Y_{j_m} X_{i_1} \ldots X_{i_{s-m}} g\right). \end{aligned} \tag{2.17}$$

The second case (2.15) leads to another Weyl–Moyal-like deformed multiplication

$$f \star g = f \exp[\hbar Y_i \otimes X_i] g. \tag{2.18}$$

Again, in the case of the canonical Poisson bracket (2.13) product (2.18) is the well-known Kupershmidt–Manin (KM) product and the deformed bracket (2.2) is the KM bracket [25, 26]. Expanding (2.18) one finds

$$f \star g = \sum_{s=0}^{\infty} \frac{\hbar^s}{s!} \left(Y_{i_1} \ldots Y_{i_s} f\right)\left(X_{i_1} \cdots X_{i_s} g\right). \tag{2.19}$$

Lemma 2.4. *The product (2.18) is associative. Moreover, it is a well-defined $\star$-product.*

Before we prove the lemma, let us introduce the product (2.18) in a little bit different notation

$$f \star g = \exp\left[\hbar Y_i^f X_i^g\right](fg) \tag{2.20}$$

where we use the symbols Y_i^f, X_i^g for vector fields acting only on f and g, respectively. The following relations for commuting differential operators X and Y are fulfilled:

$$\exp[\hbar(X + Y)] = \exp[\hbar X] \exp[\hbar Y] \tag{2.21}$$

$$\exp[\hbar XY](fg) = \exp[\hbar X(Y^f + Y^g)](fg) \tag{2.22}$$

$$\exp[\hbar XY](fg) = f \exp[\hbar (XY \otimes 1 + X \otimes Y + Y \otimes X + 1 \otimes XY)] g. \tag{2.23}$$

The first relation is standard and for the second one the proof is as follows:

$$\begin{aligned} \exp[\hbar XY](fg) &= \sum_{s=0}^{\infty} \frac{\hbar^s}{s!} X^s Y^s (fg) = \sum_{s=0}^{\infty} \frac{\hbar^s}{s!} X^s \sum_{n=0}^{s} \binom{s}{n} (Y^{s-n} f)(Y^n g) \\ &\overset{m=s-n}{=} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} X^m \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} X^n (Y^m f)(Y^n g) = \exp[\hbar X Y^f] \exp[\hbar X Y^g](fg). \end{aligned}$$

The last relation follows from the second one as

$$\begin{aligned} \exp[\hbar XY](fg) &= \exp[\hbar(X^f + X^g)(Y^f + Y^g)](fg) \\ &= \exp[\hbar(X^f Y^f + X^f Y^g + X^g Y^f + X^g Y^g)](fg). \end{aligned}$$

Proof. Using the above relations one proves the associativity of the product (2.18) as

$$\begin{aligned} (f \star g) \star h &= \left(\exp\left[\hbar Y_i^f X_i^g\right](fg)\right) \exp\left[\hbar Y_i \otimes X_i\right] h \\ &= \exp\left[\hbar Y_i^f X_i^g\right] \exp\left[\hbar (Y_i^f + Y_i^g) X_i^h\right](fgh) \\ &= \exp\left[\hbar Y_i^f \left(X_i^g + X_i^h\right)\right] \exp\left[\hbar Y_i^g X_i^h\right](fgh) \\ &= f \exp\left[\hbar Y_i \otimes X_i\right] \left(\exp\left[\hbar Y_i^g X_i^h\right](gh)\right) = f \star (g \star h). \end{aligned}$$

The rest of properties (2.1) of the $\star$-product is obvious. □

Let us define the linear automorphism $D : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$D = \exp\left[-\frac{\hbar}{2} Y_i X_i\right] \qquad D^{-1} = \exp\left[\frac{\hbar}{2} Y_i X_i\right]. \tag{2.24}$$

It relates the $\star$-product (2.18) to the product (2.16) by relation (2.8) as

$$\begin{aligned} f \star' g &= \exp\left[-\frac{\hbar}{2} Y_i X_i\right] \left(\exp\left[\frac{\hbar}{2} Y_i X_i\right](f) \exp\left[\hbar Y_i \otimes X_i\right] \exp\left[\frac{\hbar}{2} Y_i X_i\right](g)\right) \\ &= f \exp\left[-\frac{\hbar}{2}\left(Y_i X_i \otimes 1 + Y_i \otimes X_i + X_i \otimes Y_i + 1 \otimes Y_i X_i\right)\right] \\ &\quad\times \exp\left[\frac{\hbar}{2} Y_i X_i \otimes 1 + \hbar Y_i \otimes X_i + \frac{\hbar}{2} 1 \otimes Y_i X_i\right] g \\ &= f \exp\left[\frac{\hbar}{2}\left(Y_i \otimes X_i - X_i \otimes Y_i\right)\right] g = f \exp\left[\frac{\hbar}{2} Y_i \wedge X_i\right] g. \end{aligned}$$

Hence, the product (2.16) is also a well-defined $\star$-product, equivalent to the $\star$-product (2.18). Applying to (2.18)

$$D^{\alpha} = \exp\left[-\alpha \frac{\hbar}{2} Y_i X_i\right] \tag{2.25}$$

one finds infinitely many well-defined $\star$-products:

$$f \star^{\alpha} g = D^{\alpha}(D^{-\alpha} f \star D^{-\alpha} g) = f \exp\left[\frac{\hbar}{2}\left((2-\alpha) Y_i \otimes X_i - \alpha X_i \otimes Y_i\right)\right] g \tag{2.26}$$

where $\alpha \in \mathbb{R}$. All of them are equivalent to each other and all of them are quantizations of classical Poisson bracket (2.13). Note that our particular $\star$-products (2.16) and (2.18) are special cases of $\star^{\alpha}$-product (2.26) with $\alpha = 1$ and $\alpha = 0$, respectively.

Now, we impose the Lie algebra structure on the algebra $\mathcal{A}$, denoting it by $\mathcal{A}_{\alpha} = (\mathcal{A}, \star^{\alpha})$, with the commutator

$$\{f, g\}_{\star^{\alpha}} := \frac{1}{\hbar}(f \star^{\alpha} g - g \star^{\alpha} f). \tag{2.27}$$

Obviously, the automorphism (2.25) induces the isomorphisms between the Lie algebras

$$D^{\alpha'-\alpha} : \mathcal{A}_{\alpha} \longrightarrow \mathcal{A}_{\alpha'} \tag{2.28}$$

as

$$D^{\alpha'-\alpha}\{f, g\}_{\star^{\alpha}} = \{D^{\alpha'-\alpha} f, D^{\alpha'-\alpha} g\}_{\star^{\alpha'}}. \tag{2.29}$$

We will call the Lie algebras $\mathcal{A}_{\alpha}$ gauge equivalent as one can choose freely the algebra one wants to work with.

3. Poisson algebras of formal Laurent series

Consider the simplest possible case of $\dim M = 2$, when M is parametrized by a pair of coordinates (x, p). The Poisson bracket on $\mathcal{A}$ can be introduced in infinitely many ways as

$$\{f, g\}_{PB}^{r} := p^{r}\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}\right) \qquad r \in \mathbb{Z}. \tag{3.1}$$

Moreover, in $\mathcal{A}$ there exists the following Poisson subalgebra of formal Laurent series (Lax polynomials):

$$A = \left\{L = \sum_{i \in \mathbb{Z}} u_i(x) p^{i}\right\} \tag{3.2}$$

where the coefficients u_i are smooth functions of x. We assume from now on that $x \in \Omega$, where $\Omega = \mathbb{S}^1$ if u_i are periodic or $\Omega = \mathbb{R}$ if u_i belong to the Schwartz space. An appropriate symmetric product on A is given by a trace form $(a, b)_A := \mathrm{Tr}(ab)$

$$\mathrm{Tr}\, L = \int_\Omega \mathrm{res}_r L \,\mathrm{d}x \qquad \mathrm{res}_r L \equiv u_{r-1}(x) \tag{3.3}$$

which is ad-invariant [17]. In expression (3.3) the integration denotes the equivalence class of differential expressions modulo total derivatives. For a given functional $F(L) = \int_\Omega f(u)\,\mathrm{d}x$, we define its differentials as

$$\mathrm{d}F = \frac{\delta F}{\delta L} = \sum_i \frac{\delta F}{\delta u_i} p^{r-1-i} \tag{3.4}$$

where $\delta F/\delta u$ is the usual variational derivative.

We construct the simplest R-matrix, through a decomposition of A into a direct sum of Lie subalgebras. For a fixed r let

$$\begin{aligned} A_{\geqslant -r+k} &= P_{\geqslant -r+k} A = \left\{ L = \sum_{i \geqslant -r+k} u_i(x) p^i \right\} \\ A_{<-r+k} &= P_{<-r+k} A = \left\{ L = \sum_{i<-r+k} u_i(x) p^i \right\} \end{aligned} \tag{3.5}$$

where P are appropriate projections. As presented in [17], $A_{\geqslant -r+k}$, $A_{<-r+k}$ are Lie subalgebras in the following cases:

1. $k = 0 \qquad r = 0$
2. $k = 1, 2 \qquad r \in \mathbb{Z}$

which one can see through a simple inspection. Then, the R-matrix is given by the projections

$$R = \tfrac{1}{2}(P_{\geqslant -r+k} - P_{<-r+k}) = P_{\geqslant -r+k} - \tfrac{1}{2} = \tfrac{1}{2} - P_{<-r+k} \tag{3.6}$$

and its adjoint is

$$R^* = \tfrac{1}{2}(P^*_{\geqslant -r+k} - P^*_{<-r+k}) = \tfrac{1}{2} - P_{\geqslant 2r-k} = P_{<2r-k} - \tfrac{1}{2}. \tag{3.7}$$

Hence, the hierarchy of evolution equations (1.8) for Casimir functionals

$$C_n(L) = \frac{1}{n+1} \mathrm{Tr}\,(L^{n+1}) \qquad \mathrm{d}C_n(L) = L^n \tag{3.8}$$

has the form of two equivalent representations

$$L_{t_q} = \{(L^q)_{\geqslant -r+k}, L\}^r_{PB} = -\{(L^q)_{<-r+k}, L\}^r_{PB} \qquad L \in A \tag{3.9}$$

which are Lax hierarchies. Note that (3.9) are multi-Hamiltonian systems [18].

We have to explain what type of Lax operators can be used in (3.9) to obtain a consistent operator evolution equation equivalent with some nonlinear integrable dispersionless systems. We are interested in extracting closed systems for a finite number of fields. To obtain a consistent Lax equation, the Lax operator L has to form a proper submanifold of the full Poisson algebra A, i.e. the left- and right-hand sides of expression (3.9) have to coincide. They are given in the form [18]

$$k = 0 \qquad r = 0: \quad L = p^N + u_{N-2} p^{N-2} + \cdots + u_1 p + u_0 \tag{3.10}$$

$$k = 1 \qquad r \in \mathbb{Z}: \quad L = p^N + u_{N-1} p^{N-1} + \cdots + u_{1-m} p^{1-m} + u_{-m} p^{-m} \tag{3.11}$$

$$k = 2 \qquad r \in \mathbb{Z}: \quad L = u_N p^N + u_{N-1} p^{N-1} + \cdots + u_{1-m} p^{1-m} + p^{-m} \tag{3.12}$$

where the u_i are dynamical fields.

4. Weyl–Moyal-like deformation of Poisson algebras of formal Laurent series

The Poisson brackets (3.1) on $\mathcal{A}$ can be presented in the following form:

$$\{f, g\}_{PB}^{r} = f(p^r \partial_p \wedge \partial_x) g \qquad r \in \mathbb{Z}. \tag{4.1}$$

Note that this is a special case of (2.13), when $Y_1 = p^r \partial_p$ and $X_1 = \partial_x$ with $[Y_1, X_1] = 0$. For a fixed r, the Poisson bracket (4.1) on $\mathcal{A}$ can be quantized in infinitely many equivalent ways via the $\star^{\alpha}$-product (2.26)

$$f \star^{\alpha} g = f \exp\left[\frac{\hbar}{2}((2-\alpha)p^r \partial_p \otimes \partial_x - \alpha \partial_x \otimes p^r \partial_p)\right] g. \tag{4.2}$$

One finds that

$$(p^r \partial_p)^s p^m = c_s^m(r) p^{m-s(1-r)} \qquad s \in \mathbb{Z}_+ \tag{4.3}$$

where for $k \in \mathbb{Z}$

$$c_s^{k(1-r)}(r) = \begin{cases} (1-r)^s \dfrac{k!}{(k-s)!} & \text{for } k \geqslant s \text{ and } r \neq 1 \\ 0 & \text{for } s > k \geqslant 0 \text{ and } r \neq 1 \\ (-1+r)^s \dfrac{(s-k-1)!}{s!} & \text{for } k < 0 \text{ and } r \neq 1 \end{cases} \tag{4.4}$$

for $m \neq k(1-r)$

$$c_s^m(r) = m(m-(1-r)) \cdot \cdots \cdot (m-(s-1)(1-r)) \tag{4.5}$$

and for an arbitrary $m \in \mathbb{Z}$

$$c_s^m(1) = m^s. \tag{4.6}$$

One also finds the following relation, which will be useful later:

$$c_s^m(r) = (-1)^s c_s^{(s-1)(1-r)-m}(r). \tag{4.7}$$

Hence, for $\alpha \neq 0, 2$

$$up^m \star^{\alpha} vp^n = \sum_{s=0}^{\infty} \frac{\hbar^s}{2^s s!} \sum_{k=0}^{s} (-1)^k \binom{s}{k} (2-\alpha)^{s-k} \alpha^k c_{s-k}^m(r) c_k^n(r) u_{kx} v_{(s-k)x} p^{m+n-s(1-r)} \tag{4.8}$$

for $\alpha = 0$

$$up^m \star^0 vp^n = \sum_{s=0}^{\infty} \frac{\hbar^s}{s!} c_s^m(r) u v_{sx} p^{m+n-s(1-r)} \tag{4.9}$$

and for $\alpha = 2$

$$up^m \star^2 vp^n = \sum_{s=0}^{\infty} \frac{(-\hbar)^s}{s!} c_s^n(r) u_{sx} v p^{m+n-s(1-r)}. \tag{4.10}$$

Now, a simple inspection leads to the following relations: for $\alpha \neq 0, 2$

$$\begin{aligned} \{up^m, vp^n\}_{\star^{\alpha}} &= \frac{1}{\hbar}(up^m \star^{\alpha} vp^n - vp^n \star^{\alpha} up^m) \\ &= \sum_{s=0}^{\infty} \frac{\hbar^{s-1}}{2^s s!} \sum_{k=0}^{s} (-1)^k \binom{s}{k} (2-\alpha)^{s-k} \alpha^k \\ &\quad \times \left(c_{s-k}^m(r) c_k^n(r) u_{kx} v_{(s-k)x} - c_k^m(r) c_{s-k}^n(r) u_{(s-k)x} v_{kx}\right) p^{m+n-s(1-r)} \end{aligned} \tag{4.11}$$

for $\alpha = 0$

$$\{up^m, vp^n\}_{\star^0} = \sum_{s=0}^{\infty} \frac{\hbar^{s-1}}{s!} \left(c_s^m(r) u v_{sx} - c_s^n(r) u_{sx} v\right) p^{m+n-s(1-r)} \tag{4.12}$$

for $\alpha = 2$

$$\{up^m, vp^n\}_{\star^2} = \sum_{s=0}^{\infty} \frac{\hbar^{s-1}}{s!} (-1)^s \left(c_s^n(r) u_{sx} v - c_s^m(r) u v_{sx}\right) p^{m+n-s(1-r)}. \tag{4.13}$$

So, we can quantize separately the Poisson subalgebra A (3.2) to the following Lie subalgebras $A_\alpha = (A, \star^\alpha) \subset \mathcal{A}_\alpha$.

Obviously, the Lie algebras A_α for a fixed value of r are gauge equivalent under the isomorphism (2.28)

$$D^{\alpha'-\alpha} : A_\alpha \longrightarrow A_{\alpha'} \qquad D^{\alpha'-\alpha} = \exp\left[(\alpha - \alpha')\frac{\hbar}{2} p^r \partial_p \partial_x\right]. \tag{4.14}$$

Let

$$L = \sum_{m=-\infty}^{+\infty} u_m p^m \in A_\alpha \qquad L' = \sum_{n=-\infty}^{+\infty} v_n p^n \in A_{\alpha'} \tag{4.15}$$

then $L' = D^{\alpha'-\alpha} L$ and fields u_m, v_n are mutually related in the following way:

$$v_n = \sum_{s\geqslant 0} \left((\alpha - \alpha')\frac{\hbar}{2}\right)^s \frac{1}{s!} c_s^{s(1-r)+n}(r) (u_{s(1-r)+n})_{sx}. \tag{4.16}$$

Because of the gauge equivalence between the Lie algebras A_α we can choose one Lie algebra with a fixed value of α, make all necessary calculations, and then reconstruct all results for $A_{\alpha'}$ directly from the transformation (4.16).

On the other hand one can show the following relations:

$$u \star^\alpha v = uv \tag{4.17}$$

$$p^m \star^\alpha p^n = p^{m+n} \tag{4.18}$$

$$p^m \star^\alpha u = \sum_{s\geqslant 0} \frac{\hbar^s}{s!} u_{sx} \star^\alpha (p^r \partial_p)^s p^m \tag{4.19}$$

$$u \star^\alpha p^m = \sum_{s\geqslant 0} \frac{\hbar^s}{s!} (p^r \partial_p)^s p^m \star^\alpha u_{sx}. \tag{4.20}$$

As all relations (4.17)–(4.20) have the same form independently of α we skip this index in further considerations. Hence, we can quantize separately the algebra A to the following special algebra of Lax operators:

$$\mathfrak{a} = \left\{ L = \sum_{i\in\mathbb{Z}} u_i(x) \star p^i \right\}. \tag{4.21}$$

It is obviously associative algebra under commutation rules (4.19), (4.20). The algebra $\mathfrak{a}$ in the case of $r = 0$ was considered for the first time in [4]. Then, the Lie-bracket on $\mathfrak{a}$ is given by

$$\begin{aligned}\{u \star p^m, v \star p^n\}_\star &= \frac{1}{\hbar}(u \star p^m \star v \star p^n - v \star p^n \star u \star p^m) \\ &= \sum_{s=0}^{\infty} \frac{\hbar^{s-1}}{s!} [c_s^m(r) u v_{sx} - c_s^n(r) u_{sx} v] \star p^{m+n-s(1-r)}.\end{aligned} \tag{4.22}$$

Note that the algebra differs from that defined in the second section, where we introduced deformation quantization, as in (4.21) we also deformed the Lax polynomials. Let us remark that the algebra $\mathfrak{a}$ is naturally isomorphic to the algebra A_0 as $u \star^0 p^m = up^m$. Hence, in further considerations we will concentrate only on the algebra $\mathfrak{a}$, as the results for the algebras A_α for all values of α can be obtained simply by transformations (4.16) from A_0. The second reason is that $\mathfrak{a}$ can be considered as the generalization of the algebra of the pseudo-differential operators and the algebra of the shift operators in the following sense.

Let us consider the case of $r = 0$, then the rules (4.19) and (4.20) take the particular form

$$p^m \star u = \sum_{s=0} \hbar^s \binom{m}{s} u_{sx} \star p^{m-s} \tag{4.23}$$

$$u \star p^m = \sum_{s=0} (-\hbar)^s \binom{m}{s} p^{m-s} \star u_{sx} \tag{4.24}$$

and the Lie bracket (4.22) is

$$\{u \star p^m, v \star p^n\} = \sum_{s=0}^{\infty} \hbar^{s-1} \left[\binom{m}{s} u v_{sx} - \binom{n}{s} u_{sx} v \right] \star p^{m+n-s}. \tag{4.25}$$

Hence, the algebra $\mathfrak{a}$ for fixed $r = 0$ is isomorphic to the algebra of pseudo-differential operators

$$\mathfrak{g} = \left\{ \mathcal{L} = \sum_{i\in\mathbb{Z}} u_i(x) \partial_x^i \right\} \tag{4.26}$$

where the multiplication of two such operators uses the generalized Leibniz rule

$$\partial^m u = \sum_{s=0} \hbar^s \binom{m}{s} u_{sx} \partial_x^{m-s} \qquad u\partial^m = \sum_{s=0} (-\hbar)^s \binom{m}{s} \partial_x^{m-s} u_{sx} \tag{4.27}$$

where $\hbar$ is a formal parameter. The Lie algebra structure of $\mathfrak{g}$ is given by the bracket $[\mathcal{L}_1, \mathcal{L}_2] = \frac{1}{\hbar}(\mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1)$. The isomorphism is given by the function sym : $\mathfrak{g} \to \mathfrak{a}$

$$\operatorname{sym}(\mathcal{L}) = \operatorname{sym}\left(\sum_i u_i(x) \partial_x^i \right) = \sum_i u_i(x) \star p^i = L. \tag{4.28}$$

It has the important property that for arbitrary $\mathcal{L}_1, \mathcal{L}_2 \in \mathfrak{g}$

$$\operatorname{sym}(\mathcal{L}_1\mathcal{L}_2) = \operatorname{sym}(\mathcal{L}_1) \star \operatorname{sym}(\mathcal{L}_2). \tag{4.29}$$

Then it follows that

$$\operatorname{sym}([\mathcal{L}_1, \mathcal{L}_2]) = \{\operatorname{sym}(\mathcal{L}_1), \operatorname{sym}(\mathcal{L}_2)\}_\star = \{L_1, L_2\}_\star. \tag{4.30}$$

Hence, sym is the Lie algebra isomorphism. Obviously, such Lie algebras $\mathfrak{g}$ for all values of $\hbar$ are in a natural way isomorphic to the standard algebra of pseudo-differential operators ($\hbar = 1$).

Let us now consider the case of $r = 1$, then the rules (4.19) and (4.20) become

$$\begin{aligned} p^m \star u(x) &= \sum_{s=0} \frac{\hbar^s}{s!} m^s \, (u(x))_{sx} \star p^m \\ &=: \mathcal{E}^m u(x) \star p^m = u(x + m\hbar) \star p^m \end{aligned} \tag{4.31}$$

$$\begin{aligned} u(x) \star p^m &= \sum_{s=0} \frac{(-\hbar)^s}{s!} m^s p^m \star (u(x))_{sx} \\ &=: p^m \star \mathcal{E}^{-m} u(x) = p^m \star u(x - m\hbar) \end{aligned} \tag{4.32}$$

where we use the formula for Taylor expansion and consider $\mathcal{E}$ as the shift operator. The Lie bracket (4.22) is

$$\{u(x) \star p^m, v(x) \star p^n\}_\star = \frac{1}{\hbar}\left[u(x)v(x+m\hbar) - u(x+n\hbar)v(x)\right] \star p^{m+n}. \tag{4.33}$$

Hence, the algebra $\mathfrak{a}$ for a fixed $r = 1$ is isomorphic to the algebra of shift operators

$$\mathfrak{e} = \left\{\mathcal{L} = \sum_{i\in\mathbb{Z}} u_i(x)E^i\right\} \tag{4.34}$$

where E is the shift operator such that

$$E^m u(x) = u(x+m\hbar)E^m \qquad u(x)E^m = E^m u(x-m\hbar) \tag{4.35}$$

where $\hbar$ is a formal parameter. The Lie algebra structure of $\mathfrak{e}$ is given by the bracket $[\mathcal{L}_1, \mathcal{L}_2] = \frac{1}{\hbar}(\mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1)$. The isomorphism is given by the function sym $: \mathfrak{e} \to \mathfrak{a}$

$$\operatorname{sym}(\mathcal{L}) = \operatorname{sym}\left(\sum_i u_i(x)E^i\right) = \sum_i u_i(x) \star p^i = L. \tag{4.36}$$

As in the previous case, relations (4.29) and (4.30) are fulfilled for arbitrary $\mathcal{L}_1, \mathcal{L}_2 \in \mathfrak{e}$.

Let us investigate for a moment some properties of the Lie algebra $\mathfrak{a}$. The first observation is the existence of a symmetric, non-degenerate and ad-invariant product on $\mathfrak{a}$ allowing us to identify $\mathfrak{a}$ with its dual $\mathfrak{a}^*$.

Lemma 4.1. *An appropriate scalar product on* $\mathfrak{a}$ *is given by a trace form*

$$(L_1, L_2)_\mathfrak{a} := \operatorname{Tr}(L_1 \star L_2) \tag{4.37}$$

where

$$\operatorname{Tr} L = \int_\Omega \operatorname{res}_r L \,\mathrm{d}x \qquad \operatorname{res}_r L \equiv u_{r-1}(x). \tag{4.38}$$

Then (4.37) is symmetric, non-degenerate and ad-*invariant.*

Proof. The non-degeneracy of the product (4.37) is obvious. Let $L_1 = \sum_m u_m \star p^m$, $L_2 = \sum_n v_n \star p^n$, then using relations (4.19) and (4.7) we find

$$\begin{aligned}
(L_1, L_2)_\mathfrak{a} &= \operatorname{Tr}\left(\sum_{m,n} u_m \star p^m \star v_n \star p^n\right)\\
&= \operatorname{Tr}\left(\sum_{m,n}\sum_{s=0}^{\infty} \frac{\hbar^s}{s!} c_s^m(r) u_m\, (v_n)_{sx} \star p^{m+n-s(1-r)}\right)\\
&= \int_\Omega \sum_n \sum_{s=0}^{\infty} \frac{\hbar^s}{s!} c_s^{(s-1)(1-r)-n}(r) u_{(s-1)(1-r)-n}\, (v_n)_{sx}\,\mathrm{d}x\\
&= \int_\Omega \sum_n \sum_{s=0}^{\infty} \frac{\hbar^s}{s!} (-1)^s c_s^{(s-1)(1-r)-n}(r) (u_{(s-1)(1-r)-n})_{sx} v_n\,\mathrm{d}x\\
&= \int_\Omega \sum_n \sum_{s=0}^{\infty} \frac{\hbar^s}{s!} c_s^n(r) (u_{(s-1)(1-r)-n})_{sx} v_n\,\mathrm{d}x\\
&= \operatorname{Tr}\left(\sum_{m,n}\sum_{s=0}^{\infty} \frac{\hbar^s}{s!} c_s^n(r)\, (u_m)_{sx}\, v_n \star p^{m+n-s(1-r)}\right)\\
&= \operatorname{Tr}\left(\sum_{m,n} v_n \star p^n \star u_m \star p^m\right) = (L_2, L_1)_\mathfrak{a}
\end{aligned}$$

where we have used the integration by parts. The ad-invariance follows from associativity of $\star$-product and symmetry of the product (4.37) as

$$
\begin{aligned}
(\{A, B\}_\star, C)_{\mathfrak{a}} &= \mathrm{Tr}\left(\frac{1}{\hbar}(A \star B \star C - B \star A \star C)\right) \\
&= \mathrm{Tr}\left(\frac{1}{\hbar}(B \star C \star A - C \star B \star A)\right) = (\{B, C\}_\star, A)_{\mathfrak{a}}. \qquad \square
\end{aligned}
$$

As a consequence, for operators $L = \sum_i u_i \star p^i$, the vector fields $\frac{\mathrm{d}}{\mathrm{d}t} L \equiv L_t$ and differentials $\mathrm{d}H(L)$ are conveniently parametrized by

$$L_t = \sum_i (u_i)_t \star p^i \qquad \mathrm{d}H(L) = \frac{\delta H}{\delta L} = \sum_i p^{r-1-i} \star \frac{\delta H}{\delta u_i} \tag{4.39}$$

where $\partial H/\partial u_i$ is the usual variational derivative of a functional $H = \int_\Omega h(u, u_x, \ldots)\,\mathrm{d}x$. In these frames the trace duality assumes the usual Euclidean form

$$(\mathrm{d}H, L_t)_{\mathfrak{a}} = \mathrm{Tr}\,(\mathrm{d}H \star L_t) = \sum_i \int_\Omega \frac{\delta H}{\delta u_i} (u_i)_t\,\mathrm{d}x. \tag{4.40}$$

Now, one can simply rewrite the trace formula from $\mathfrak{a}$ to A_0 as $\mathfrak{a} \cong A_0$. Then, the appropriate trace formulae on A_α for Lax polynomials $L = \sum_n u_n p^n$ are given by

$$\mathrm{Tr}\,L = \int_\Omega \mathrm{res}_r L\,\mathrm{d}x \qquad \mathrm{res}_r L \equiv u_{r-1}(x) \tag{4.41}$$

which are well defined, as the trace formula is invariant under transformations (4.14) since from (4.16) it follows that $v_{r-1} = u_{r-1}$. Hence, the scalar products take the form

$$(L_1, L_2)_{A_\alpha} := \mathrm{Tr}\,(L_1 \star^\alpha L_2) \tag{4.42}$$

and differentials $\mathrm{d}H(L)$ are conveniently parametrized by

$$\mathrm{d}H(L) = \frac{\delta H}{\delta L} = \sum_n p^{r-1-n} \star^\alpha \frac{\delta H}{\delta u_n}. \tag{4.43}$$

Note that in formulae (4.41)–(4.43) one has to use the explicit form of $\star^\alpha$-products.

5. R-matrix formalism and Lax hierarchies for Lie algebra $\mathfrak{a}$

To construct the integrable field systems one has to split the algebra $\mathfrak{a}$ into a direct sum of Lie subalgebras. Observing (4.22) one finds that in general it can be done only for $r = 0$ or $r = 1$. Let us remark that it is possible to choose a Lie subalgebra of $\mathfrak{a}$ in the form

$$\left\{ L = \sum_{i \in \mathbb{Z}} u_i(x) \star p^{i(1-r)} \right\} \qquad r \neq 1 \tag{5.1}$$

which can be further split into a direct sum of Lie subalgebras, but this case is simply related by the transformation $p' = p^{1-r}$, $x' = x/(1-r)$ to the algebra $\mathfrak{a}$ for the case of $r = 0$.

Now, we decompose $\mathfrak{a}$ for $r = 0, 1$ into a direct sum of Lie subalgebras in the following way. Let

$$
\begin{aligned}
\mathfrak{a}_{\geqslant -r+k} &= P_{\geqslant -r+k}\mathfrak{a} = \left\{ L = \sum_{i \geqslant -r+k} u_i(x) \star p^i \right\} \\
\mathfrak{a}_{< -r+k} &= P_{< -r+k}\mathfrak{a} = \left\{ L = \sum_{i < -r+k} u_i(x) \star p^i \right\}
\end{aligned}
\tag{5.2}
$$

where P are appropriate projections. Then, $\mathfrak{a}_{\geqslant -r+k}$, $\mathfrak{a}_{<-r+k}$ are Lie subalgebras in the case of $r = 0$ for $k = 0, 1, 2$ and in the case of $r = 1$ for $k = 1, 2$. Hence, the R-matrix is given by the projections

$$R = \tfrac{1}{2}(P_{\geqslant -r+k} - P_{<-r+k}) = P_{\geqslant -r+k} - \tfrac{1}{2} = \tfrac{1}{2} - P_{<-r+k} \tag{5.3}$$

and its adjoint is

$$R^* = \tfrac{1}{2}(P^*_{\geqslant -r+k} - P^*_{<-r+k}) = \tfrac{1}{2} - P_{\geqslant 2r-k} = P_{<2r-k} - \tfrac{1}{2}. \tag{5.4}$$

The hierarchy of evolution equations is generated by the Casimir functionals

$$C_n(L) = \frac{1}{n+1}\mathrm{Tr}\,(L^{n+1}) \qquad \mathrm{d}C_n(L) = L^n \qquad L^n = L \star L \star \ldots \star L \tag{5.5}$$

and for appropriate k has the form of two equivalent representations

$$L_{t_n} = \{(L^n)_{\geqslant -r+k}, L\}_\star = -\{(L^q)_{<-r+k}, L\}_\star \tag{5.6}$$

which are Lax hierarchies.

The Lie algebras A_α can be decomposed into a direct sum of Lie subalgebras in exactly the same way as $A_0 \cong \mathfrak{a}$. Hence, the R-matrix (5.3) is invariant under transformations (4.14). Moreover, as transformations (4.14) are Lie algebra isomorphisms (2.29) the Lax hierarchies (5.6) are also invariant with respect to them.

For constructing (1+1)-dimensional closed systems with a finite number of fields we have to choose properly restricted Lax operators L which give consistent Lax equations (5.6). To obtain a consistent Lax equation, the Lax operator L has to form a proper submanifold of the full Poisson algebra under consideration, i.e. the left- and right-hand sides of expression (5.6) have to lie inside this submanifold. In the case of $r = 0$ the admissible simplest restricted Lax operators are given in the form

$$k = 0: \quad L = p^N + u_{N-2} \star p^{N-2} + \cdots + u_1 \star p + u_0 \tag{5.7}$$

$$k = 1: \quad L = p^N + u_{N-1} \star p^{N-1} + \cdots + u_0 + p^{-1} \star u_{-1} \tag{5.8}$$

$$k = 2: \quad L = u_N \star p^N + u_{N-1} \star p^{N-1} + \cdots + u_0 + p^{-1} \star u_{-1} + p^{-2} \star u_{-2}. \tag{5.9}$$

In the case of $r = 1$ the admissible simplest restricted Lax operators are

$$k = 1: \quad L = p^N + u_{N-1} \star p^{N-1} + \cdots + u_{1-m} \star p^{1-m} + u_{-m} \star p^{-m} \tag{5.10}$$

$$k = 2: \quad L = u_N \star p^N + u_{N-1} \star p^{N-1} + \cdots + u_{1-m} \star p^{1-m} + p^{-m}. \tag{5.11}$$

We will now compare the Lax operators related to soliton systems with the Lax operators related to the dispersionless systems. As follows, the class of operators (5.7) is the same as the class of dispersionless operators (3.10). Hence, all dispersionless systems for $r = 0$ and $k = 0$ have counterpart soliton systems. For $r = 0$ and $k = 1, 2$ the classes of dispersionless Lax operators are wider. The operators (5.8) by the quasi-classical limit ($\hbar \to 0$) reduce to the operators (3.11) for $m = 1$. The operators (5.9) reduce to (3.12) for $m = 2$ but the field u_{-2} by the quasi-classical limit becomes time independent. For $r = 1$ the classes of operators (5.10), (5.11) and (3.11), (3.12) are the same, respectively. Thus, all of them have the counterpart lattice field systems. The remaining dispersionless systems for $r \neq 0, 1$ and some for $r = 0$ do not have counterpart soliton systems in the quantization scheme considered.

The evolution systems (5.6), with the Casimir functionals (5.5) as Hamiltonian functions, are tri-Hamiltonian

$$L_{t_n} = \theta_1(L)\,\mathrm{d}C_n = \tfrac{1}{2}\theta_2(L)\,\mathrm{d}C_{n-1} = \theta_3(L)\,\mathrm{d}C_{n-2} \tag{5.12}$$

as it was for the algebra of pseudo-differential operators and the algebra of shift operators. Nevertheless, as we work with restrictions (5.7)–(5.11), a reduction procedure for the Hamiltonian structures of the general representations (5.12) will be necessary.

The case of $r = 0$. All Lax operators (5.7)–(5.9) form a proper submanifold with respect to the linear Poisson tensor (1.9) which is given for R-matrix (5.3) in two equivalent representations

$$\begin{aligned}\theta_1(L)\mathrm{d}H &= \{(\mathrm{d}H)_{\geqslant k}, L\}_\star - (\{\mathrm{d}H, L\}_\star)_{\geqslant -k}\\ &= -\{(\mathrm{d}H)_{<k}, L\}_\star + (\{\mathrm{d}H, L\}_\star)_{<-k} \qquad k = 0, 1, 2.\end{aligned} \tag{5.13}$$

Since $(\partial H/\partial L)_{\geqslant k} = 0$ for $k = 0, 1$, the linear Poisson tensor for these cases is given in simpler form

$$\theta_1\left(\frac{\delta H}{\delta L}\right) = \left(\left\{L, \frac{\delta H}{\delta L}\right\}_\star\right)_{\geqslant -k}. \tag{5.14}$$

The quadratic bracket for $k = 0$ is given by (1.12)

$$\begin{aligned}\theta_2(L)\mathrm{d}H &= \hbar\{L, (\{\mathrm{d}H, L\}_\star^+)_{\geqslant 0}\}_\star - \hbar\{L, (\{\mathrm{d}H, L\}_\star)_{\geqslant 0}\}_\star^+\\ &= -\hbar\{L, (\{\mathrm{d}H, L\}_\star^+)_{<0}\}_\star + \hbar\{L, (\{\mathrm{d}H, L\}_\star)_{<0}\}_\star^+\end{aligned} \tag{5.15}$$

and can be properly restricted to a subspace of the form

$$L = p^N + u_{N-1} \star p^{N-1} + u_{N-2} \star p^{N-2} + \cdots + u_1 \star p + u_0. \tag{5.16}$$

Hence a Dirac reduction $u_{N-1} = 0$ is required, with the final result

$$\theta_2^{\mathrm{red}}\left(\frac{\delta H}{\delta L}\right) = \frac{1}{\hbar}\left[\left(L \star \frac{\delta H}{\delta L}\right)_{\geqslant 0} \star L - L \star \left(\frac{\delta H}{\delta L} \star L\right)_{\geqslant 0}\right] + \frac{\hbar}{N}\left\{\partial_x^{-1}\mathrm{res}\left\{\frac{\delta H}{\delta L}, L\right\}_\star, L\right\}_\star \tag{5.17}$$

where $\theta_2^{\mathrm{red}}(L)$ is compatible with the linear one (5.14). For $k = 1$, the quadratic tensor $\theta_2(L)$ is given by (1.10)

$$\theta_2(L)\,\mathrm{d}H = \frac{1}{\hbar}[A_1(L \star \mathrm{d}H) \star L - L \star A_2(\mathrm{d}H \star L) + S(\mathrm{d}H \star L) \star L - L \star S^*(L \star \mathrm{d}H)] \tag{5.18}$$

where

$$\begin{aligned}&A_1(b) = b_{\geqslant 1} - b_0 + b_{-1} - b_{<-1} - 2\partial_x^{-1}\mathrm{res}\, b \qquad b \in \mathfrak{a}\\ &A_2(b) = b_{\geqslant 0} - b_{<0} + 2\partial_x^{-1}\mathrm{res}\, b\\ &S(b) = -2b_{-1} + 2\partial_x^{-1}\mathrm{res}\, b \qquad S^*(b) = -2b_0 - 2\partial_x^{-1}\mathrm{res}\, b\end{aligned} \tag{5.19}$$

satisfy (1.11). The Poisson tensor (5.18) admits a proper restriction to Lax operators of the form (5.8). Hence, we have

$$\begin{aligned}\theta_2\left(\frac{\delta H}{\delta L}\right) = \frac{1}{\hbar}&\left[\left(L \star \frac{\delta H}{\delta L}\right)_{\geqslant 1} \star L - L \star \left(\frac{\delta H}{\delta L} \star L\right)_{\geqslant 0} + L \star \left(L \star \frac{\delta H}{\delta L}\right)_0\right]\\ &- \partial_x^{-1}\mathrm{res}\left(\left\{\frac{\delta H}{\delta L}, L\right\}_\star\right) \star L + \hbar\left\{\partial_x^{-1}\mathrm{res}\left\{\frac{\delta H}{\delta L}, L\right\}_\star, L\right\}_\star.\end{aligned} \tag{5.20}$$

For $k = 2$, in contrast to the previous cases, we still do not know the proper form of the quadratic tensor θ_2.

The restricted Lax operators (5.7)–(5.9) do not form proper submanifolds with respect to the cubic Poisson tensor (1.15)

$$\begin{aligned}\theta_3(L)\,\mathrm{d}H &= \{(L \star \mathrm{d}H \star L)_{\geqslant k}, L\}_\star - L \star (\{\mathrm{d}H, L\}_\star)_{\geqslant -k} \star L\\ &= -\{(L \star \mathrm{d}H \star L)_{<k}, L\}_\star + L \star (\{\mathrm{d}H, L\}_\star)_{<k} \star L \qquad k = 0, 1, 2.\end{aligned} \tag{5.21}$$

Nevertheless, the Dirac reduction can be applied. Here, in contrast to the previous cases, the number of constraints depends on N, so the reduction has to be considered separately for each N.

The case of $r = 1$. Both Lax operators (5.10) and (5.11) form a proper submanifold with respect to the linear Poisson tensor (1.9)

$$\begin{aligned}\theta_1(L)\,\mathrm{d}H &= \{(\mathrm{d}H)_{\geqslant -1+k}, L\}_\star - (\{\mathrm{d}H, L\}_\star)_{\geqslant 2-k}\\ &= -\{(\mathrm{d}H)_{<-1+k}, L\}_\star + (\{\mathrm{d}H, L\}_\star)_{<2-k} \qquad k = 1, 2.\end{aligned} \tag{5.22}$$

Hence, no additional restrictions are needed.

The quadratic tensor is given by the special case (1.12), nevertheless the Lax operators (5.10), (5.11) do not form a proper submanifold. Actually, the proper submanifold is

$$L = u_N \star p^N + u_{N-1} \star p^{N-1} + \cdots + u_{1-m} \star p^{1-m} + u_{-m} \star p^{-m}. \tag{5.23}$$

Thus for $k = 1$ the Dirac constraint $u_N = 1$ gives

$$\begin{aligned}\theta_2^{\mathrm{red}}\left(\frac{\delta H}{\delta L}\right) &= \hbar\left\{\left(\left\{L, \frac{\delta H}{\delta L}\right\}_\star^+\right)_{\geqslant 0}, L\right\}_\star - \hbar\left\{L, \left(\left\{\frac{\delta H}{\delta L}, L\right\}_\star\right)_{\geqslant 1}\right\}_\star^+\\ &\quad + \hbar\left\{(1+\mathcal{E}^{-N})(1-\mathcal{E}^{-N})^{-1}\mathrm{res}\left\{\frac{\delta H}{\delta L}, L\right\}_\star, L\right\}_\star\end{aligned} \tag{5.24}$$

and for $k = 2$ with Dirac constraint $u_{-m} = 1$ we get

$$\begin{aligned}\theta_2^{\mathrm{red}}\left(\frac{\delta H}{\delta L}\right) &= \hbar\left\{\left(\left\{L, \frac{\delta H}{\delta L}\right\}_\star^+\right)_{\geqslant 1}, L\right\}_\star - \hbar\left\{L, \left(\left\{\frac{\delta H}{\delta L}, L\right\}_\star\right)_{\geqslant 0}\right\}_\star^+\\ &\quad - \hbar\left\{(1+\mathcal{E}^{m})(1-\mathcal{E}^{m})^{-1}\mathrm{res}\left\{\frac{\delta H}{\delta L}, L\right\}_\star, L\right\}_\star.\end{aligned} \tag{5.25}$$

The restricted Lax operators (5.10), (5.11) do not form proper submanifolds with respect to the cubic Poisson tensor (1.15)

$$\begin{aligned}\theta_3(L)\,\mathrm{d}H &= \{(L \star \mathrm{d}H \star L)_{\geqslant -1+k}, L\}_\star - L \star (\{\mathrm{d}H, L\}_\star)_{\geqslant 2-k} \star L\\ &= -\{(L \star \mathrm{d}H \star L)_{<-1+k}, L\}_\star + L \star (\{\mathrm{d}H, L\}_\star)_{<2-k} \star L \qquad k = 1, 2.\end{aligned} \tag{5.26}$$

Nevertheless, the Dirac reduction can be applied. Again, the number of constraints depends on N, so the reduction has to be considered separately for each N.

Let us now consider more precisely the transformations from the evolution systems constructed from the algebra $\mathfrak{a}$ to the systems constructed from A_α for $r = 0$ and $r = 1$. The linear transformation $D^\alpha : A_0 \longrightarrow A_\alpha$ is simply given by (4.14) as $\mathfrak{a} \cong A_0$. First consider the case of $r = 0$. Let

$$\begin{aligned}L &= \sum_{m\geqslant 0} u_m p^m + \sum_{m<0} p^m \star^0 u_m \in A_0\\ L_\alpha &= \sum_{n\geqslant 0} w_n p^n + \sum_{n<0} p^n \star^\alpha w_n \in A_\alpha.\end{aligned} \tag{5.27}$$

Then, $L_\alpha = D^\alpha L$, where $D^\alpha = \exp\left(-\frac{1}{2}\alpha\hbar\partial_p\partial_x\right)$. As follows the dynamical fields are interrelated in the following way for $n \geqslant 0$:

$$\begin{aligned}w_n &= \sum_{s\geqslant 0}\left(-\frac{\alpha}{2}\hbar\right)^s \binom{n+s}{s}(u_{n+s})_{sx}\\ u_n &= \sum_{s\geqslant 0}\left(\frac{\alpha}{2}\hbar\right)^s \binom{n+s}{s}(w_{n+s})_{sx}\end{aligned} \tag{5.28}$$

and for $n < 0$ $u_n = w_n$. We denote this transformation in the operator form by $w = \phi(u)$, then the Fréchet derivative of ϕ, such that $(w_m)_t = \sum_n (\phi')^n_m \, (u_n)_t$, is

$$(\phi')^n_m = \sum_{k\geqslant 0} \frac{\partial w_m}{\partial (u_n)_{kx}} \partial_x^k = \begin{cases} \left(-\frac{\alpha}{2}\hbar\right)^{n-m} \binom{n}{n-m} \partial_x^{n-m} & \text{for } n \geqslant m \geqslant 0 \\ \delta_{m,n} & \text{for } m < 0 \\ 0 & \text{for the rest} \end{cases} \tag{5.29}$$

and its adjoint is

$$(\phi'^{\dagger})^m_n = \sum_{k\geqslant 0} (-1)^k \partial_x^k \frac{\partial w_m}{\partial (u_n)_{kx}} = \begin{cases} \left(\frac{\alpha}{2}\hbar\right)^{n-m} \binom{n}{n-m} \partial_x^{n-m} & \text{for } n \geqslant m \geqslant 0 \\ \delta_{m,n} & \text{for } m < 0 \\ 0 & \text{for the rest.} \end{cases} \tag{5.30}$$

Consider now the case of $r = 1$. Let

$$L = \sum_m u_m p^m \in A_0 \qquad L_\alpha = \sum_n w_n p^n \in A_\alpha. \tag{5.31}$$

Then, $L_\alpha = D^\alpha L$, where $D^\alpha = \exp\left(-\frac{1}{2}\alpha\hbar p \partial_p \partial_x\right)$ and from (4.16) the relations between the dynamical fields are

$$\begin{aligned} w_n(x) &= \sum_{s\geqslant 0} \left(-\frac{\alpha}{2}\hbar\right)^s \frac{1}{s!} n^s \, (u_n(x))_{sx} = \mathcal{E}^{-n\alpha/2} u_n(x) = u_n\left(x - n\frac{\alpha}{2}\hbar\right) \\ u_n(x) &= \sum_{s\geqslant 0} \left(\frac{\alpha}{2}\hbar\right)^s \frac{1}{s!} n^s \, (w_n(x))_{sx} = \mathcal{E}^{-n\alpha/2} w_n(x) = w_n\left(x + n\frac{\alpha}{2}\hbar\right). \end{aligned} \tag{5.32}$$

Again, if we denote the transformation as $w = \phi(u)$, then

$$(\phi')^n_m = \delta_{m,n} \mathcal{E}^{-m\alpha/2} \qquad \text{and} \qquad (\phi'^{\dagger})^n_m = \delta_{m,n} \mathcal{E}^{m\alpha/2}. \tag{5.33}$$

Thus, obviously in both cases, when

$$u_t = \theta \, \mathrm{d}H \qquad w_t = \tilde{\theta} \, \mathrm{d}\tilde{H} \tag{5.34}$$

then

$$w_t = \phi' u_t \qquad \tilde{\theta} = \phi \theta \phi'^{\dagger} \qquad \mathrm{d}H = \phi'^{\dagger} \, \mathrm{d}\tilde{H}. \tag{5.35}$$

We will now display examples of some field and lattice soliton systems calculated in the quantization scheme considered. We consider the Lax hierarchy (5.6) with little changed numerations of evolution variables

$$L_{t_n} = \{(L^{n/N})_{\geqslant -r+k}, L\}_\star \tag{5.36}$$

where N is the highest order of the Lax operator L. We will exhibit the first non-trivial equation of the Lax hierarchy (5.36). For simplicity we present only the bi-Hamiltonian structure. The advantage of the use of $\mathfrak{a}$ algebra is that during whole calculations there is no need to use the $\star^\alpha$-product in explicit form and we only use the commutation relations (4.19), (4.20). As a result, one gets the same equations and Poisson structures as these obtained from quantized algebra A_0. The Hamiltonian systems related to quantized algebras A_α are simply reconstructed via the linear transformation (5.28), (5.32) and formulae (5.35). Such a procedure of calculations is applied in the present examples and we have written only the final results for A_α.

Example 5.1. *The Boussinesq system*: $r = 0, k = 0$.

The dispersionless Boussinesq Hamiltonian systems are given in the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x \\ -\frac{2}{3}uu_x \end{pmatrix} = \theta_1 \mathrm{d}H_1 = \theta_2^{\mathrm{red}}\,\mathrm{d}H_2 \tag{5.37}$$

where the Poisson tensors are

$$\theta_1 = 3\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \qquad \theta_2^{\mathrm{red}} = \begin{pmatrix} \partial_x u + u\partial_x & 2\partial_x v + v\partial_x \\ \partial_x v + 2v\partial_x & -\frac{2}{3}u\partial_x u \end{pmatrix} \tag{5.38}$$

and

$$H_1 = \frac{1}{3}\int_\Omega \left(-\frac{1}{9}u^3 + v^2\right)\mathrm{d}x \qquad H_2 = \int_\Omega v\,\mathrm{d}x. \tag{5.39}$$

System (5.37) has the following Lax representation [18]:

$$L_{t_2} = \left\{\left(L^{2/3}\right)_{\geqslant 0}, L\right\}^0_{PB} \tag{5.40}$$

for the Lax operator in the form

$$L = p^3 + up + v. \tag{5.41}$$

The quantization procedure leads now to the following Lax operator in $\mathfrak{a}$:

$$L = p^3 + u \star p + v. \tag{5.42}$$

Then, one can derive the Boussinesq system from

$$L_{t_2} = \left\{\left(L^{2/3}\right)_{\geqslant 0}, L\right\}_\star. \tag{5.43}$$

Now, by the transformation to the algebras A_α one finds the following systems:

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} &= \begin{pmatrix} 2v_x + (\alpha - 1)\,\hbar u_{2x} \\ -\frac{2}{3}uu_x + (1-\alpha)\,\hbar v_{2x} - \left(\frac{\alpha^2}{2} - \alpha + \frac{2}{3}\right)\hbar^2 u_{3x} \end{pmatrix} \\ &= \theta_1\,\mathrm{d}H_1 = \theta_2\,\mathrm{d}H_2. \end{aligned}\tag{5.44}$$

The respective Poisson tensors can be calculated from (5.14)

$$\theta_1 = 3\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{5.45}$$

and from (5.17)

$$\theta_2^{\mathrm{red}} = \frac{1}{2}\begin{pmatrix} \theta_{uu} & \theta_{uv} \\ -(\theta^{uv})^\dagger & \theta_{vv} \end{pmatrix} \tag{5.46}$$

where

$$\theta_{uu} = \partial_x u + u\partial_x + 2\hbar^2\partial_x^3$$

$$\theta_{uv} = 2\partial_x v + v\partial_x + \hbar\left[\alpha\partial_x u_x + \alpha u_x\partial_x - \partial_x^2 u + \alpha u\partial_x^2\right] + (\alpha - 1)\,\hbar^3\partial_x^4$$

$$\begin{aligned}\theta_{vv} = {}&-\frac{2}{3}u\partial_x u + (1-\alpha)\,\hbar\left[\partial_x^2 v - v\partial_x^2\right] - \left(\frac{\alpha^2}{4} - \frac{\alpha}{2} + \frac{2}{3}\right)\hbar^2\left[\partial_x^3 u + u\partial_x^3\right] \\ &- \frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right)\hbar^2\left[\partial_x^2 u_x - u_x\partial_x^2\right] - \left(\frac{\alpha^2}{2} - \alpha + \frac{2}{3}\right)\hbar^4\partial_x^5.\end{aligned}$$

The Hamiltonians are given in the following form:

$$H_1 = \frac{1}{3}\int_\Omega \left[-\frac{1}{9}u^3 + v^2 + (\alpha - 1)\,\hbar u_x v + \frac{\alpha^2}{4}\hbar^2 u_x^2 + \left(\frac{\alpha}{2} - \frac{1}{3}\right)\hbar^2 uu_{2x}\right]\mathrm{d}x \tag{5.47}$$

$$H_2 = \int_\Omega v\,\mathrm{d}x. \tag{5.48}$$

In the case of $\alpha = 0$ (5.44) is obviously the standard case of a Boussinesq system obtained from Gel'fand–Dikii hierarchy, for $\alpha = 1$ it is the Moyal case. The limit $\hbar \to 0$ of (5.44) gives (5.37) as it should.

Example 5.2. *The Kaup–Broer (KB) system*: $r = 0, k = 1$.

The dispersionless system is given by

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = 2\begin{pmatrix} uu_x + v_x \\ u_x v + u v_x \end{pmatrix} = \theta_1\,\mathrm{d}H_1 = \theta_2^{\mathrm{red}}\,\mathrm{d}H_2 \tag{5.49}$$

where

$$\begin{aligned} \theta_1 &= \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} & \theta_2^{\mathrm{red}} &= \begin{pmatrix} 2\partial_x & \partial_x u \\ u\partial_x & \partial_x v + v\partial_x \end{pmatrix} \\ H_1 &= \int_\Omega (u^2 v + v^2)\,\mathrm{d}x & H_2 &= \int_\Omega uv\,\mathrm{d}x \end{aligned} \tag{5.50}$$

is known as the Benney system. The Lax representation for (5.49) is [18]

$$L_{t_2} = \{(L^2)_{\geqslant 0}, L\}_{PB}^0 \tag{5.51}$$

where

$$L = p + u + vp^{-1}. \tag{5.52}$$

The quantized Lax operator in $\mathfrak{a}$ is

$$L = p + u + p^{-1} \star v. \tag{5.53}$$

We derive the KB system, which is the dispersive Benney system, from

$$L_{t_2} = \{(L^2)_{\geqslant 0}, L\}_\star. \tag{5.54}$$

And, by the transformation to the algebras A_α one gets the following systems:

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} &= \begin{pmatrix} 2uu_x + (\alpha+1)\,\hbar u_{2x} + 2v_x \\ 2\,(uv)_x - \alpha\left(1 - \frac{\alpha}{2}\right)\hbar^2 u_{3x} - (\alpha+1)\hbar v_{2x} \end{pmatrix} \\ &= \theta_1\,\mathrm{d}H_1 = \theta_2\,\mathrm{d}H_2. \end{aligned} \tag{5.55}$$

The Poisson tensors are

$$\theta_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{5.56}$$

and

$$\theta_2 = \frac{1}{2}\begin{pmatrix} \partial_x & \partial_x u + (\alpha+1)\,\hbar\partial_x^2 \\ u\partial_x - (\alpha+1)\partial_x^2 & v\partial_x + \partial_x v + \frac{1}{2}\alpha\hbar\left(u\partial_x^2 - \partial_x^2 u\right) - \alpha\left(1 + \frac{\alpha}{2}\right)\hbar^2\partial_x^3 \end{pmatrix}. \tag{5.57}$$

The Hamiltonians are

$$H_1 = \int_\Omega \left[u^2 v + v^2 - (\alpha+1)\,uv_x + \frac{\alpha}{2}\hbar u^2 u_x + \frac{\alpha^2}{4}\hbar^2 u_x^2\right]\mathrm{d}x \tag{5.58}$$

$$H_2 = \int_\Omega \left[uv + \frac{\alpha}{2}\hbar u u_x\right]\mathrm{d}x. \tag{5.59}$$

For $\alpha = 0$ (5.55) is the standard case of the KB system and for $\alpha = 1$ it is the Moyal case.

Example 5.3. *Toda system*: $r = 1, k = 1$.

The dispersionless Toda system has the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} v_x \\ u_x v \end{pmatrix} = \theta_1 \,\mathrm{d}H_1 = \theta_2^{\mathrm{red}} \,\mathrm{d}H_2 \tag{5.60}$$

where

$$\begin{aligned} &\theta_1 = \begin{pmatrix} 0 & \partial_x v \\ v\partial_x & 0 \end{pmatrix} && \pi_0^{\mathrm{red}} = \begin{pmatrix} \partial_x v + v\partial_x & u\partial_x v \\ v\partial_x u & 2v\partial_x v \end{pmatrix} \\ &H_1 = \frac{1}{2}\int_\Omega (u^2 + 2v)\,\mathrm{d}x && H_2 = \int_\Omega u\,\mathrm{d}x. \end{aligned} \tag{5.61}$$

The Lax representation for (5.60) is [18]

$$L_{t_2} = \{(L^2)_{\geqslant 0}, L\}^1_{PB} \tag{5.62}$$

where the Lax operator is

$$L = p + u + vp^{-1}. \tag{5.63}$$

Then, the quantization scheme leads to the following Lax operator in $\mathfrak{a}$ in the form:

$$L = p + u(x) + v(x) \star p^{-1}. \tag{5.64}$$

One derives the Toda system from

$$L_{t_2} = \{(L^2)_{\geqslant 0}, L\}_\star. \tag{5.65}$$

Next, by the transformation to the algebras A_α one finds the following systems:

$$\begin{aligned} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_{t_2} &= \frac{1}{\hbar}\begin{pmatrix} v\left(x + \left(1 - \frac{\alpha}{2}\right)\hbar\right) - v\left(x - \frac{\alpha}{2}\hbar\right) \\ v(x)\left[u\left(x + \frac{\alpha}{2}\hbar\right) - u\left(x - \left(1 - \frac{\alpha}{2}\right)\hbar\right)\right] \end{pmatrix} \\ &= \theta_1\,\mathrm{d}H_1 = \theta_2\,\mathrm{d}H_2. \end{aligned} \tag{5.66}$$

The respective Poisson tensors are

$$\theta_1 = \frac{1}{\hbar}\begin{pmatrix} 0 & \left[\mathcal{E}^{(1-\alpha/2)} - \mathcal{E}^{-\alpha/2}\right]v(x) \\ v(x)\left[\mathcal{E}^{\alpha/2} - \mathcal{E}^{-(1-\alpha/2)}\right] & 0 \end{pmatrix} \tag{5.67}$$

and

$$\theta_2^{\mathrm{red}} = \frac{1}{\hbar}\begin{pmatrix} \mathcal{E}^{(1-\alpha/2)}v(x)\mathcal{E}^{\alpha/2} - \mathcal{E}^{-\alpha/2}v(x)\mathcal{E}^{(\alpha/2-1)} & u(x)\left[\mathcal{E}^{(1-\alpha/2)} - \mathcal{E}^{-\alpha/2}\right]v(x) \\ v(x)\left[\mathcal{E}^{\alpha/2} - \mathcal{E}^{(\alpha/2-1)}\right]u(x) & v(x)[\mathcal{E} - \mathcal{E}^{-1}]v(x) \end{pmatrix}. \tag{5.68}$$

The Hamiltonians are

$$H_1 = \int_\Omega \left[v\,(x) + \frac{1}{2}u^2(x)\right]\mathrm{d}x \qquad H_2 = \int_\Omega u(x)\,\mathrm{d}x. \tag{5.69}$$

The case of $\alpha = 0$ of (5.66) is the standard case of the Toda system, the case of $\alpha = 1$ is the Moyal case. Note that in our construction the Toda equation depends on the continuous coordinate x in contrast to the standard case when x is an integer.

Example 5.4. *Three field system*: $r = 1, k = 2$.

The dispersionless system is given in the form

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_2} = \begin{pmatrix} 2uw_x \\ u_x + vw_x \\ v_x \end{pmatrix} = \theta_1\,\mathrm{d}H_1 = \theta_2^{\mathrm{red}}\,\mathrm{d}H_2 \tag{5.70}$$

where the Poisson tensors are

$$\theta_1 = 3\begin{pmatrix} 0 & 0 & 2u\partial_x \\ 0 & \partial_x u + u\partial_x & v\partial_x \\ 2\partial_x u & \partial_x v & 0 \end{pmatrix}$$
$$\theta_2^{\mathrm{red}} = \begin{pmatrix} 6u\partial_x u & 4u\partial_x v & 2u\partial_x w \\ 4v\partial_x u & 2v\partial_x v + u\partial_x w + w\partial_x u & v\partial_x w + \partial_x u + 2u\partial_x \\ 2w\partial_x u & w\partial_x v + 2\partial_x u + u\partial_x & \partial_x v + v\partial_x \end{pmatrix} \tag{5.71}$$

and

$$H_1 = \int_\Omega \left(v + \frac{1}{2}w^2\right) \mathrm{d}x \qquad H_2 = \int_\Omega w\,\mathrm{d}x. \tag{5.72}$$

System (5.70) has the following Lax representation:

$$L_{t_2} = \{(L)_{\geqslant 1}, L\}_{PB}^1 \tag{5.73}$$

for the Lax operator in the form

$$L = up^2 + vp + w + p^{-2}. \tag{5.74}$$

The quantization procedure leads to the following Lax operator in $\mathfrak{a}$:

$$L = u(x) \star p^2 + v(x) \star p + w(x) + p^{-2}. \tag{5.75}$$

Then, one derives the dispersive version of (5.70) from

$$L_{t_2} = \{(L)_{\geqslant 1}, L\}_\star \tag{5.76}$$

and by the transformation to the algebras A_α one finds the following systems:

$$\begin{aligned} u(x)_{t_2} &= \frac{1}{\hbar}u(x)\,[w\,(x + (2-\alpha)\hbar) - w\,(x - \alpha\hbar)] \\ v(x)_{t_2} &= \frac{1}{\hbar}\Big[u\left(x + \frac{\alpha}{2}\hbar\right) - u\left(x + \left(\frac{\alpha}{2} - 1\right)\hbar\right) \\ &\qquad + v(x)\Big[w\left(x - \left(\frac{\alpha}{2} - 1\right)\hbar\right) - w\left(x - \frac{\alpha}{2}\hbar\right)\Big]\Big] \\ w(x)_{t_2} &= \frac{1}{\hbar}\left[v\left(x + \frac{\alpha}{2}\hbar\right) - v\left(x + \left(\frac{\alpha}{2} - 1\right)\hbar\right)\right]. \end{aligned} \tag{5.77}$$

The linear Poisson tensor is

$$\theta_1 = \frac{1}{\hbar}\begin{pmatrix} 0 & 0 & \theta_1^{uw} \\ 0 & \theta_1^{vv} & \theta_1^{vw} \\ -\left(\theta_1^{uw}\right)^\dagger & -\left(\theta_1^{vw}\right)^\dagger & 0 \end{pmatrix} \tag{5.78}$$

where

$$\begin{aligned} \theta_1^{uw} &= u(x)[\mathcal{E}^{(2-\alpha)} - \mathcal{E}^{-\alpha}] \\ \theta_1^{vv} &= \mathcal{E}^{\alpha/2}u(x)\mathcal{E}^{(1-\alpha/2)} - \mathcal{E}^{(\alpha/2-1)}u(x)\mathcal{E}^{-\alpha/2} \\ \theta_1^{vw} &= v(x)\big[\mathcal{E}^{(1-\alpha/2)} - \mathcal{E}^{-\alpha/2}\big]. \end{aligned}$$

The quadratic Poisson tensor is

$$\theta_2^{\mathrm{red}} = \frac{1}{\hbar}\begin{pmatrix} \theta_2^{uu} & \theta_2^{uv} & \theta_2^{uw} \\ -\left(\theta_2^{uv}\right)^\dagger & \theta_2^{vv} & \theta_2^{vw} \\ -\left(\theta_2^{uw}\right)^\dagger & -\left(\theta_2^{vw}\right)^\dagger & \theta_2^{ww} \end{pmatrix} \tag{5.79}$$

where

$$\theta_2^{uu} = u(x)[\mathcal{E}^2 + \mathcal{E} - \mathcal{E}^{-1} - \mathcal{E}^{-2}]u(x)$$
$$\theta_2^{uv} = u(x)\big[\mathcal{E}^{(2-\alpha/2)} + \mathcal{E}^{(1-\alpha/2)} - \mathcal{E}^{-\alpha/2} - \mathcal{E}^{-(1+\alpha/2)}\big]v(x)$$
$$\theta_2^{uw} = u(x)[\mathcal{E}^{(2-\alpha)} - \mathcal{E}^{-\alpha}]w(x)$$
$$\theta_2^{vv} = v(x)[\mathcal{E} - \mathcal{E}^{-1}]v(x) + \mathcal{E}^{\alpha/2}u(x)\mathcal{E}^{(1-\alpha)}w(x)\mathcal{E}^{\alpha/2} - \mathcal{E}^{-\alpha/2}w(x)\mathcal{E}^{(\alpha-1)}u(x)\mathcal{E}^{-\alpha/2}$$
$$\theta_2^{vw} = v(x)\big[\mathcal{E}^{(1-\alpha/2)} - \mathcal{E}^{-\alpha/2}\big]w(x) + \mathcal{E}^{\alpha/2}u(x)\mathcal{E}^{(2-\alpha)} - \mathcal{E}^{(\alpha/2-1)}u(x)\mathcal{E}^{-\alpha}$$
$$\theta_2^{ww} = \mathcal{E}^{\alpha/2}v(x)\mathcal{E}^{(1-\alpha/2)} - \mathcal{E}^{(\alpha/2-1)}v(x)\mathcal{E}^{-\alpha/2}.$$

The respective Hamiltonians have the form

$$H_1 = \int_\Omega \left[v(x) + \frac{1}{2}w(x)^2\right]\mathrm{d}x \qquad H_2 = \int_\Omega w(x)\mathrm{d}x. \tag{5.80}$$

The case with $\alpha = 0$ ($\hbar = 1$) and integer x was constructed in [13].

6. Conclusions

In this paper we have presented a systematic construction of the field and lattice soliton systems from underlying multi-Hamiltonian dispersionless systems. Actually, the passage has been made on the level of appropriate Lax representations through the Weyl–Moyal-like deformation quantization procedure. In a previous paper [18], we have constructed Lax representations for a wide class of dispersionless systems with multi-Hamiltonian structures, derived from classical R-matrix theory. The number of constructed dispersionless systems is much greater than the number of known soliton systems (dispersive integrable systems). So, the question arises whether for any dispersionless Lax hierarchy one can construct a related soliton hierarchy. We have tried to obtain an answer to this question via the procedure of deformation quantization for Poisson algebras of dispersionless systems and appropriate R-matrix theory. We have managed to quantize all Poisson algebras (with arbitrary r (3.1)) but the R-matrix, at least of the form (1.17), exists only in the case of two algebras, i.e. for $r = 0$ and $r = 1$, respectively. The first case leads to soliton field systems related to the algebra of pseudo-differential operators (4.26), and the second one leads to lattice soliton systems related to the algebra of shift operators (4.34). In that sense, although we have failed to construct new soliton equations through the deformation procedure presented, nevertheless we have found a unified procedure for the construction of field and lattice Hamiltonian soliton systems in one scheme.

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